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A Note on Hilbert Series of Free Metabelian Lie Algebras

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Abstract: Let F_m be the free metabelian Lie algebra of finite rank. In the present work we propose an elementary proof in computation of Hilbert series of F_m.

Keywords: Free metabelian Lie algebras, Hilbert Series.

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1. INTRODUCTION

In 19th century the concept of Hilbert (or Poincaré) series was introduced in terms of finitely generated commutative algebras over a field K, where the generating series of the Hilbert function are related with K linear dimension of the graded or filtered algebra. In 1890 Hilbert proved how the sum of Hilbert series is always a rational function allowing a finite description of such an invariant and is known as a fundamental property of a finitely generated commutative algebra. In 1927 Francis Sowerby Macaulay proved how the Hilbert series of a commutative algebra is equal to the series of a corresponding monomial algebra, that is an algebra whose generators are related by monomials and this monomial is defined by leading monomial ideal with respect to a suitable monomial ordering. For more see [1,2,3,4,5].

The polynomial algebra $K[X_m]$ is free in the class of all commutative algebras. Since the Hilbert series of polynomial algebra has structure of finitely generated commutative algebras then is a rational function for more see in [6,7,8] together with ideas of classical invariant theory in [9,10]. For more historical background, see the chapter 1-2 of [11] and [12,13] for the basic facts regarding Lie algebras, varieties of associative in [14] and Chapter 1 of [15] regarding algebraic geometry.

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The work of rational symmetric functions related with Hilbert series in [10] were used in [16,17] where they applied the method of them in order to work in algebra of constants of free metabelian Lie algebras. The categories of metabelian associative and metabelian Lie algebras are interesting and many of researchers have been paying attention among of them are [18, 19, 20] on algebraic geometry of free metabelian Lie algebra, [21] on Gröbner-Shirshov bases for metabelian Lie algebras, [22,23,24,25] on endomorphisms and automorphisms of free metabelian Lie algebras, [26,27] on the Backer-Campbell-Hausdorff formula for free metabelian Lie algebras and [28,29] on the categories and classification of metabelian group and Lie algebras respectively.

In the present work, we come out with an elementary proof of formula of Hilbert series of the free metabelian Lie algebras which appears also in [13,30,31] with a proof using graduate techniques in [13].

2. PRELIMINARIES

For a graded algebra A over a field K, recall that the formal power series

$$H(A,z)=\sum_{n\geq 0}\dim_K(A^{(n)})\,z^n$$

is called the Hilbert (or Poincaré) series of A, where $A^{(n)}$ is the subspace of A containing the elements of homogeneous degree n. For instance, considering the commutative polynomial algebra $K[x_1,x_2,\dots,x_m]$ and non-commutative polynomial algebra $K\langle x_1,x_2,\dots,x_m\rangle$ in m variables, we have the followings.

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$$H(K[x_1,x_2,\dots,x_m],z)=\frac{1}{(1-z)^m}$$
$$H(K\langle x_1,x_2,\dots,x_m\rangle,z)=\frac{1}{1-mz}.$$

Let L_m be the free Lie algebra over the field K of characteristic 0 generated by a finite set $X=\{x_1,\cdots,x_m\}$ with $m\geq 2$. Now we define the quotient algebra

$$F_m={L_m}/{L''_m}={L_m}/{[[L_m,L_m],[L_m,L_m]]}$$

this is the free metabelian (solvable of length 2) Lie algebra of rank m defined by metabelian identity $[[t_1,t_2],[t_3,t_4]]=0$. F_m is generated by free generators $y_1=x_1+L''_m,\cdots,y_m=$ *xm+Lm″* and this algebra is of a basis consisting of *y1,...,ym* together with all left normed monomials of the form:

$$[y_{i_1},y_{i_2},\dots,y_{i_k}]=\left[\cdots\left[[y_{i_1},y_{i_2}],y_{i_3}\right],\dots,y_{i_k}\right],\ i_1<i_2\geq i_3\geq\cdots\geq i_k.$$

Let $w\in F'_m$ and let $x,y\in F_m$ be arbitrary elements. Then as a consequence of Jacobi identity we have

$$[w,x,y]=[w,y,x].$$

Thus, F'_m is furnished with a natural structure of module of $K[y_1,\dots,y_m]$.

The following proposition is well known:

Proposition 2.1: Consider equation $n_1+n_2+\cdots+n_r=n$, for $r\in\mathbb{Z}^+$, $n,n_i\geq 0$ and $1\leq i\leq r$. Then number of distinct nonnegative integer solutions satisfying the equation is

$$\binom{n+r-1}{r-1}.$$

From the Proposition 2.1 we can obtain the number of basis of free metabelian Lie algebra of rank $m\geq 2$.
Let $\mathfrak{B}_{m,n}$ indicate the canonical basis for the homogeneous subspace of degree n in the free metabelian Lie algebra F_m, and let $\eta_{m,n}$ denote the number of elements in $\mathfrak{B}_{m,n}$ which coincides with $\dim(F_m)^{(n)}$. For example, consider the free metabelian Lie algebra $F_2=K\langle y_1,y_2\rangle$ with 2 generators. $\mathfrak{B}_{2,1}=\{y_1,y_2\}$ and $\eta_{2,1}=2$, while $\mathfrak{B}_{2,2}=\{[y_1,y_2]\}$ and $\eta_{2,2}=1$. In general, $\mathfrak{B}_{2,n}$ contains the elements of the form

$$[y_1,y_2,y_2,\dots,y_2,y_1,\dots,y_1]$$

of length n. Thus $n-2$ positions are filled by y_2s followed by y_1s after $[y_1,y_2]$. Hence the number $\eta_{2,n}$ is equal to the number of nonnegative solutions of equation

$$n_1+n_2=n-2$$

where n_1 indicates the number of y_1s and n_2 stands for the number of y_2s used in those $n-2$ positions after $[y_1,y_2]$. Hence by above proposition we have

$$\eta_{2,n}=\binom{(n-2)+2-1}{2-1}=\binom{n-1}{1}=n-1$$

Lemma 2.2: Dimension $\eta_{m,n}$ of $F_m^{(n)}$ is

$$\eta_{m,n}=\binom{n-1}{1}+2\binom{n}{2}+\cdots+(m-1)\binom{n+m-3}{m-1}$$

Proof: Let us prove this statement by induction on number of generator $m\geq 2$. As illustrated before, it is straightforward to check the formula for $m=2$. Now let

$$\eta_{m-1,n}=\binom{n-1}{1}+\cdots+(m-2)\binom{n+m-4}{m-2}$$

and let us explicitly write down the elements of $\mathfrak{B}_{m,n}$. Firstly, it is a direct result that

$$\mathfrak{B}_{m,n}=B_2\cup B_3\cup\cdots\cup B_{m-1}\cup B_m$$

where the subset B_j is of elements of the form

$$[\_,y_j,\underbrace{p_1,\cdots,p_{n-2}}_{n-2\text{ position}}]$$

thus p_is can be chosen from the set $\{y_1,\cdots,y_j\}$. The first place allows $y_1,\cdots,y_{j-1}$ and number of elements in B_j is $(j-1)$ times number of nonnegative solutions of equation

$$n_1+\cdots+n_j=n-2$$

such that n_k indicates number of y_ks used for those $n-2$ places. One must observe from this point that

$$|\mathfrak{B}_{m-1,n}|=\eta_{m-1,n}=|B_2\cup B_3\cup\cdots\cup B_{m-1}|$$

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hence it is sufficient to show that

$$|B_m| = (m-1)\binom{n+m-3}{m-1}$$

the elements of B_m are in the form

$$[\_\, y_m, \underbrace{p_1, \cdots, p_{n-2}}_{n-2 \text{ position}}]$$

thus p_is can be chosen from the set $\{y_1, \cdots, y_m\}$. The first place allows $y_1, \cdots, y_{m-1}$ and number of elements in B_m is $(m-1)$ times number of nonnegative solutions of equation

$$n_1 + \cdots + n_m = n - 2$$

such that n_k indicates number of y_ks used for those $n-2$ places. By proposition 2.1, the number is

$$|B_m| = (m-1)\binom{(n-2)+m-1}{m-1}$$

which completes the proof.

3. MAIN RESULTS

In this section we provide the main results of our objective.

Lemma 3.1: Hilbert Series of $F_2 = K\langle y_1, y_2\rangle$ is

$$H(F_2, z) = 1 + 2z + \frac{2z-1}{(1-z)^2}.$$

Proof: Let $F_2 = K\langle y_1, y_2\rangle$ be the free metabelian Lie algebra with generators y_1 and y_2. We know that number of elements in homogeneous subspace of degree n is equal to $\eta_{2,n} = n-1$ for $n \geq 2$. The Hilbert Series of F_2 comes out from this formula:

$$H(F_2, z) = \sum_{n=0}^{\infty} (\dim F_2^{(n)}) \cdot z^n = 0 + 2 \cdot z^1 + \sum_{n=2}^{\infty} \eta_{2,n} \cdot z^n$$

since $\dim F_2^{(0)} = 0$ and $\dim F_2^{(1)} = 2$. Hence, we have

$$H(F_2, z) = 2z + \sum_{n=2}^{\infty} (n-1) z^n$$

$$= 2z + \sum_{n=2}^{\infty} n\, z^n - \sum_{n=2}^{\infty} z^n = 2z + Q - P$$

where P and Q are the power series so,

$$P = \sum_{n=2}^{\infty} z^n \quad \text{and} \quad Q = \sum_{n=2}^{\infty} n\, z^n.$$

We are going to solve this accordingly:

For $P = \sum_{n=2}^{\infty} z^n$, from the polynomial algebra we have

$$H(K[y_1], z) = \sum_{n=0}^{\infty} \dim K[y_1]^{(n)}\, z^n$$

$$= \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots = \frac{1}{1-z}.$$

Recall that in writing the rational formula on right side of equality, we assume that z is a variable lying in the set of real numbers such that the series converges. Thus, we can have derivative in computations.

Now, for P we have $n \geq 2$ therefore our equation becomes

$$1 + z + \sum_{n=2}^{\infty} z^n = \frac{1}{1-z}$$

$$1 + z + P = \frac{1}{1-z}$$

and hence

$$P = \sum_{n=2}^{\infty} z^n = \frac{1}{1-z} - (1+z).$$

Now, for Q we have,

$$Q = \sum_{n=2}^{\infty} n\, z^n = 2z^2 + 3z^3 + \cdots = z(2z + 3z^2 + \cdots)$$

$$= z\left(\sum_{n=2}^{\infty} n\, z^{n-1}\right).$$

It is seen that the parenthesis is derivative of P, so we obtain,

$$Q = z \cdot P' = z \cdot \left(\sum_{n=2}^{\infty} z^n\right)' = z \cdot \left[\frac{1}{1-z} - (1+z)\right]'$$

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$$= z \cdot \left[\frac{1'(1-z) - 1(1-z)'}{(1-z)^2} - 1 \right] = z \cdot \left(\frac{1}{(1-z)^2} - 1 \right).$$

Hence, we have

$$Q = \sum_{n=2}^{\infty} n\, z^n = \frac{z}{(1-z)^2} - z.$$

Let come back on our Hilbert Series

$$H(F_2, z) = 2z + Q - P$$

$$= 2z + \frac{z}{(1-z)^2} - z - \left[\frac{1}{1-z} - (1+z) \right]$$

$$= 1 + 2z + \frac{2z-1}{(1-z)^2}.$$

Therefore, the Hilbert Series of free metabelian Lie algebra F_2 of two generators is:

$$H(F_2, z) = 1 + 2z + \frac{2z-1}{(1-z)^2}.$$

Theorem 3.2: For $m \geq 3$, we have that

$$H(F_m, z) = H(F_{m-1}, z) + z + \frac{(m-1)z^2}{(1-z)^m}.$$

Proof: From the direct computation on above Lemma 3.1 we have

$$H(F_3, z) - H(F_2, z) = 1 + 3z + \frac{3z-1}{(1-z)^3} - \left[1 + 2z + \frac{2z-1}{(1-z)^2} \right]$$

$$= 1 + 3z - 1 - 2z + \frac{3z-1}{(1-z)^3} - \frac{2z-1}{(1-z)^2}$$

$$= z + \frac{3z - 1 - (2z-1)(1-z)}{(1-z)^3}$$

$$= z + \frac{3z - 1 - (2z - 2z^2 - 1 + z)}{(1-z)^3} = z + \frac{2z^2}{(1-z)^3}.$$

and,

$$H(F_4, z) - H(F_3, z) = 1 + 4z + \frac{4z-1}{(1-z)^4} - \left[1 + 3z + \frac{3z-1}{(1-z)^3} \right]$$

$$= 1 + 4z - 1 - 3z + \frac{4z-1}{(1-z)^4} - \frac{3z-1}{(1-z)^3}$$

$$= z + \frac{4z - 1 - (3z-1)(1-z)}{(1-z)^4}$$

$$= z + \frac{4z - 1 - (3z - 3z^2 - 1 + z)}{(1-z)^4} = z + \frac{3z^2}{(1-z)^4}.$$

Thus, the formula holds for $m = 3,4$.

On the other side, we obtain that

$$H(K[Y_m], z) = \sum_{r \geq 0} \dim K[Y_m]^{(r)} z^r = \frac{1}{(1-z)^m}$$

where $Y_m = \{y_1, \ldots, y_m\}$.

For the dimension of $K[Y_m]^{(r)}$ we need to know the canonical forms of basis elements in $\mathfrak{B}_{K[Y_m]^{(r)}}$. They are in the form

$$y_1^{n_1} y_2^{n_2} \cdots y_m^{n_m}$$

such that $n_1, n_2, \ldots, n_m \geq 0$, $n_1 + \cdots + n_m = r$. By the Lemma 3.1, the number of such elements is

$$\binom{r+m-1}{m-1}.$$

Hence, we have the new expression for $H(K[Y_m], z)$ as follows:

$$H(K[Y_m], z) = \frac{1}{(1-z)^m} = \sum_{r=0}^{\infty} \dim K[Y_m]^{(r)} z^r$$

$$= \sum_{r=0}^{\infty} \binom{r+m-1}{m-1} z^r.$$

Let us multiply the following equality by z^2

$$\sum_{r=0}^{\infty} \binom{r+m-1}{m-1} z^r = \frac{1}{(1-z)^m}$$

to get

$$\sum_{r=0}^{\infty} \binom{r+m-1}{m-1} z^{r+2} = \frac{z^2}{(1-z)^m}.$$

Let $n = r + 2$, and rewrite the above formula:

$$\sum_{n=2}^{\infty} \binom{(n-2)+m-1}{m-1} z^n = \frac{z^2}{(1-z)^m}$$

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$$\Rightarrow \sum_{n=2}^{\infty} \binom{n+m-3}{m-1} z^n = \frac{z^2}{(1-z)^m}$$

by multiplying $(m-1)$ and adding z on both sides we obtain

$$z + \sum_{n=2}^{\infty} (m-1) \binom{n+m-3}{m-1} z^n = z + \frac{(m-1)z^2}{(1-z)^m}.$$

Recall that, by Lemma 2.2,

$$\eta_{m,n} = \binom{n-1}{1} + 2\binom{n}{2} + \cdots + (m-1)\binom{n+m-3}{m-1}$$

giving the numbers of element in $\mathfrak{B}_{m,n}$, hence

$$\eta_{m,n} - \eta_{m-1,n} = (m-1)\binom{n+m-3}{m-1}.$$

Rewriting the previous equation, we have

$$mz - (m-1)z + \sum_{n=2}^{\infty} (\eta_{m,n} - \eta_{m-1,n})\, z^n = z + \frac{(m-1)z^2}{(1-z)^m}$$

$$\Rightarrow \left(mz + \sum_{n=2}^{\infty} \eta_{m,n}\, z^n \right) - \left((m-1)z + \sum_{n=2}^{\infty} \eta_{m-1,n}\, z^n \right)$$

$$= z + \frac{(m-1)z^2}{(1-z)^m}$$

$$H(F_m, z) - H(F_{m-1}, z) = z + \frac{(m-1)z^2}{(1-z)^m}.$$

Therefore, we have

$$H(F_m, z) = H(F_{m-1}, z) + z + \frac{(m-1)z^2}{(1-z)^m}$$

which completes the proof.

The following formula can be found in [13,31], with a proof a proof using young diagrams in [13]. Throughout the paper we use completely undergraduate linear algebra techniques.

Corollary 3.3: The Hilbert series of the free metabelian Lie algebra F_m is

$$H(F_m, z) = 1 + mz + \frac{mz-1}{(1-z)^m}$$

Proof: Clearly, the formula holds for $m = 2$ by Lemma 3.1. Assume that the formula is true for $m-1$ as the induction hypothesis:

$$H(F_{m-1}, z) = 1 + (m-1)z + \frac{(m-1)z-1}{(1-z)^{m-1}}.$$

By the Theorem 3.2,

$$H(F_m, z) = H(F_{m-1}, z) + z + \frac{(m-1)z^2}{(1-z)^m}$$

$$= 1 + (m-1)z + \frac{(m-1)z-1}{(1-z)^{m-1}} + z + \frac{(m-1)z^2}{(1-z)^m}$$

$$= 1 + mz + \frac{z(m-1+1)-1}{(1-z)^m}$$

therefore, for all $m \geq 2$

$$H(F_m, z) = 1 + mz + \frac{mz-1}{(1-z)^m}$$

this completes the proof.

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